2021 Online Physics Olympiad: Invitational Contest

Theory Solutions

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Instructions for Theoretical Exam

The theoretical examination consists of 4 long answer questions and 100 points over 2 full days from August 13, 0:01 am GMT.

- The team leader should submit their final solution document in this google form. We don't anticipate a tie, but in the rare circumstance that there is one, the time you submit will be used to break it.

- If you wish to request a clarification, please use this form. To see all clarifications, view this document.

- Each question in this examination are equally worth 25 points. Be sure to spend your time wisely.

- Participants are given a google form where they are allowed to submit up-to 1 gigabyte of data for their solutions. It is recommended that participants write their solutions in LaTeX. However, handwritten solutions (or a combination of both) are accepted too. If participants have more than one photo of a handwritten solution (`jpg, png`, etc), it is required to organize them in the correct order in a `pdf` before submitting. If you wish a premade LaTeX template, we have made one for you here.

- Since each question is a long answer response, participants will be judged on the quality of your work. To receive full points, participants need to show their work, including deriving equations. As a general rule of thumb, any common equations (such as the ones in the IPhO formula sheet) can be cited without proof.

- Remember to state any approximations made and which system of equations were solved after every step. Explicitly showing every step of algebra is not necessary. Participants may leave all final answers in symbolic form (in terms of variables) unless otherwise specified. Be sure to state all assumptions.

Problems

- **T1: Levitation** by Ashmit Dutta

- **T2: Thomas Precession** by Jacob Nie

- **T3: Moving Media** by Eddie Chen

- **T4: Missing Energy** by Kushal Thaman

List of Constants

Proton mass	$m_p = 1.67 \cdot 10^{-27}$ kg
Neutron mass	$m_n = 1.67 \cdot 10^{-27}$ kg
Electron mass	$m_e = 9.11 \cdot 10^{-31}$ kg
Avogadro's constant	$N_0 = 6.02 \cdot 10^{23}$ mol^{-1}
Universal gas constant	$R = 8.31$ J/(mol $\cdot$ K)
Boltzmann's constant	$k_B = 1.38 \cdot 10^{-23}$ J/K
Electron charge magnitude	$e = 1.60 \cdot 10^{-19}$
1 electron volt	1 eV $= 1.60 \cdot 10^{-19}$ J
Speed of light	$c = 3.00 \cdot 10^{8}$ m/s
Universal Gravitational constant	$G = 6.67 \cdot 10^{-11}$ (N $\cdot$ m^2)/kg^2
Acceleration due to gravity	$g = 9.81$ m/s^2
1 unified atomic mass unit	1 u $= 1.66 \cdot 10^{-27}$ kg $= 931$ MeV/c^2
Planck's constant	$h = 6.63 \cdot 10^{-34}$ J $\cdot$ s $= 4.41 \cdot 10^{-15}$ eV $\cdot$ s
Permittivity of free space	$\epsilon_0 = 8.85 \cdot 10^{-12}$ C^2/(N $\cdot$ m^2)
Coulomb's law constant	$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^{9}$ (N $\cdot$ m^2)/C^2
Permeability of free space	$\mu_0 = 4\pi \cdot 10^{-7}$ T $\cdot$ m/A
Stefan-Boltzmann constant	$\sigma = 5.67 \cdot 10^{-8}$ W/m^2/K^4

T1: Levitation

Levitation is a widely researched area in physics with wide applications in real life. Commercial high speed trains use magnetic levitation to transport passengers and the very physics of aerodynamics helps planes fly in the sky. Although the many uses of levitation physics apply on macroscopic dimensions, this problem will also analyze the various applications of levitation on the microscopic scale.

Optical Tweezers

An optical tweezer (OT) is a device which uses tightly focused laser beams to trap an object in all three spacial dimensions ($x - y - z$). Consider using an OT to trap a dielectric polystyrene nanosphere with mass m, radius R, and relative dielectric constant ε_r. A laser beam is directed to the vertical z-direction (the laser beam is similar to a monochromatic light wave propagating a sparse medium.) The laser beam has a wavelength λ. The time averaged intensity in the z-direction due to light can be considered to follow a Gaussian distribution function:

$$I(\rho, z) = I_0 \left(\frac{W_0}{W(z)}\right)^2 \exp\left(\frac{-2\rho^2}{W(z)^2}\right)$$

where ρ is the distance from the center of the beam and W_0 is known as the waist size, or the measure of the beam size at the point of its focus. Here the waist length, in general, follows $W(z) = W_0\sqrt{1 + z^2/z_R^2}$ where $z_R = \pi W_0^2/\lambda$ denotes the Rayleigh length.

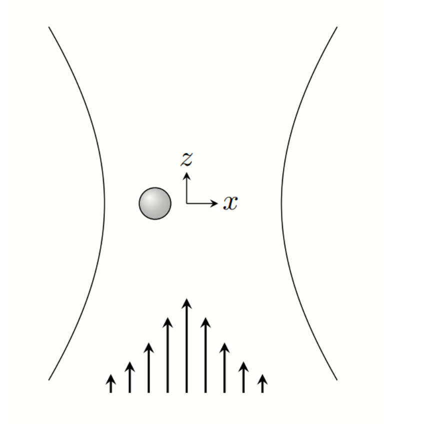


Figure 1: A nanosphere placed off center in a Gaussian beam.

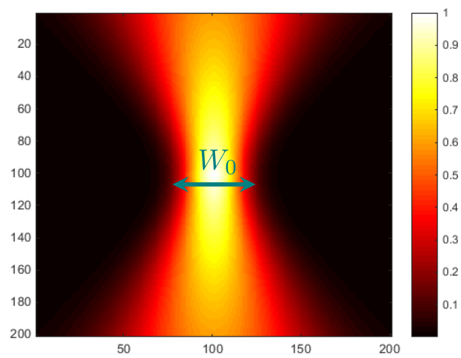


Figure 2: A graphic of the intensity distribution in a Gaussian beam.

The OT traps particles via three different forces: **scattering forces** created by the change in momentum of light scattered or absorbed by a particle; **gradient forces** due to the polarization of the particle created by the strong electric fields of the laser beam; and, **radiation forces** produced by an accelerating charge. The total power radiated by an oscillating electric dipole with dipole moment p_0 at frequency ω will be $P_R = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$, where c is the speed of light.

1. **(6 pts.)** In the Rayleigh spectrum, the nanosphere's size is such that $\lambda \gg R$. Find the oscillation frequency Ω and equilibrium position of the nanosphere when slightly displaced a distance $d \ll W_0$ in the x-direction.[1]

In the Mie spectrum, the particle size is no longer negligible such as in part A, and follows $R \gtrsim \lambda$[2]. As a result, a non-homogenous electric field is incident upon the sphere and scattering forces are no longer neglectable. Parts B and C will investigate the nanosphere in the Mie spectrum.

[1] When the exam was administered, it was said that to neglect the scattering forces in this part. In reality, the radiation forces are a type of scattering force as it takes light and re-emits it. For ambiguity, the equilibrium position was not considered in grading.

[2] The order of magnitude of R is greater than λ.

2. **(5 pts.)** Determine the scattering force and torque on the nanosphere as a function of a distance $x \ll W_0$ from its origin. We can consider $x \sim R$. Assume that 100% of light is transmitted for simplicity.[3] The index of refraction of the nanosphere is n while the index of refraction of the medium is m. You may express your answer as an integral if needed.

3. **(2 pts.)** In a simplified model, the torque acting on the nanosphere can be represented as $\boldsymbol{\tau} = \kappa\boldsymbol{\omega}$ where κ is a numerical constant and $\boldsymbol{\omega}$ is the angular velocity of the nanosphere. At $t = 0$, the nanosphere is stationary with an angular velocity ω_0 and the OT is turned on. After a time T, the OT is turned off with the nanosphere left floating in the medium again. Determine the angular velocity of the nanosphere after a time t passes where t is the time of the entire process. It can be expressed as a piecewise function.

Acoustic Levitation

Objects can also be trapped via sound waves. Consider the simplest way to model sound waves; that is, in one dimension. A cylindrical tube of length L_0 with ambient temperature and pressure T_0 and P_0 is fixed with a piston at one end[4] of cross-sectional area S that moves periodically as $x(t) = A\cos(2\pi f t)$ where $A \ll L_0$ is the amplitude of the piston and f is the frequency of the process. The tube contains n monatomic particles of mass m per unit volume. As the piston moves back and forth, the air in the tube compresses or expands (rarefies) in the tube as a sound wave. The molecules within the tube move back or forth parallel to their equilibrium position as the air travels within the piston. Neglect any viscous or turbulent friction within the pipe.

4. **(6 pts.)** What is the average power required to move the piston? Consider the limits of $f \gg c_s/A$ and $f \ll c_s/A$ where c_s is the speed of the sound wave.

In sound waves, the density perturbations are very small, so it can be assumed that $\Delta\rho \ll \rho_0$[5]. where ρ_0 is the original density of the pipe. Furthermore, the wavelength of the sound waves are much larger than the mean free path of the gas molecules. The sound wave created by the oscillating piston moves at a speed c_s and dissipates the power given by the piston.

5. **(4 pts.)** As a result of compression, the air within the sound wave has a larger temperature $\Delta T \ll T_0$. Find the change in temperature ΔT. If you were unable to solve problem 4, express the average power to move the piston as P.

6. **(2 pts.)** Consider a small cylindrical object of radius $R < \sqrt{S}$ and width $h \ll R$ in the pipe where variations of pressure on the cylinder's surface are negligible. Determine the force F acting on the cylinder when the sound wave passes through it. If the pipe is placed on a vertical plane where gravity is present, qualitatively describe what location(s) the cylinder would levitate.

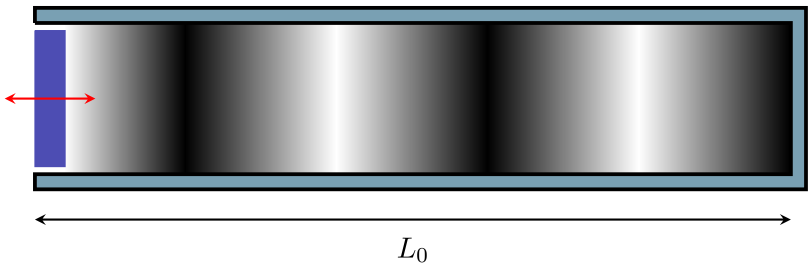


Figure 3: A visualization of how acoustic waves within the pipe are created via the oscillation of the piston.

[3] In reality, some light will be reflected due to Fresnel's equations.
[4] In some models, the piston can represent the moving cone of a loud speaker.
[5] You cannot use $\Delta\rho$ as a variable in this problem, but you can use ρ_0.

Solution

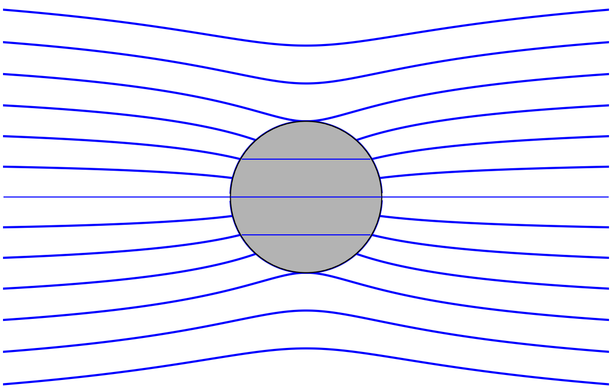
(a) The Rayleigh approximation tells us that $\lambda \gg R$. Thus, the electric field around the microsphere is homogenous; hence, the polarization of the microsphere will be as well. Furthermore, since the nanoball is displaced a small distance $d \ll W_0$, the intensity distribution can be approximated by Taylor expanding $e^x \approx 1 + x$. In other words:

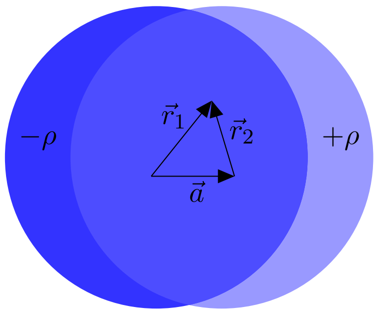
$$I(x) \approx I_0 \left(1 - \frac{2x^2}{W_0^2}\right).$$

Intensity and electric field can be related via the time averaged Poynting vector as as

$$I = |\langle \vec{S} \rangle| = \frac{1}{\mu_0} |\langle \vec{E} \times \vec{B} \rangle| = \frac{1}{c\mu_0} \frac{E_{\mathrm{m}}^2}{2}$$

where E_{m} is the maximum electric field of the Gaussian beam. As the nanoball becomes polarized, it obtains a dipole moment of $\vec{p} = -\alpha \vec{E}$. It is required to find the value of α of a dielectric ball. The polarized sphere can be divided into two different polarized rods of charge densities $+\rho$ and $-\rho$ separated a distance a, replacing them by patches of charge (circles) that are superimposed. Positive charges started accumulating on the upper half of the sphere while negative charges start accumulating on the lower half of the sphere as the electric field redistributes the charges.





Thus, the field outside the small sphere is the sum of E and the field due to an electric dipole that is at the center of the nanoball. The dipole moment expressed in terms of its polarization is

$$\vec{p} = \sum_i q\vec{r}_i = \vec{P} \cdot V = \frac{4}{3}\pi R^3 \vec{P}.$$

The total field at some point outside of the sphere is no longer uniform in the neighborhood of the sphere and is now the sum of the field E and the field generated through the polarized matter, i.e. $\vec{E} = \vec{E}_0 + \vec{E}_{\mathrm{dipole}}$. Thus, the field depends on the polarization vector of the dielectric which can be expressed in terms of its electric susceptibility as

$$\vec{P} = \chi_{e0} \vec{E}_{\mathrm{in}} = ({}_r - 1)_0 \vec{E}_{\mathrm{in}}.$$

Note that the potential at all the points on the spherical boundary is simply the dipole potential or

$$= \frac{p\cos\theta}{4\pi_0 R^2} = \frac{PR\cos\theta}{3_0}.$$

Substituting $R\cos\theta = z$ implies $= Pz/3_0$. By uniqueness theorem, this must be the only solution which means that $E_z = \frac{\partial}{\partial z} = -P/3_0$. Hence, the internal field of the sphere follows $\vec{E}_{\mathrm{in}} = -\vec{P}/3_0$. Superposition yields

$$\vec{E}_{\mathrm{in}} = \vec{E}_0 - \frac{{}_r - 1}{3}\vec{E}_{\mathrm{in}} \implies \vec{E}_{\mathrm{in}} = \frac{3}{2 + {}_r}\vec{E}.$$

Inserting this field into the dipole momentum tells us that

$$\vec{p} = \frac{4}{3}\pi R^3(_r - 1)_0 \frac{3}{2+_r}\vec{E} \implies \alpha = \frac{4\pi R^3(_r - 1)_0}{2+_r}.$$

The gradient force on the ball depends on the gradients of the various components of its fields. In other words, $\vec{F}(t) = (\vec{p}\cdot\nabla)\vec{E}(t)$. Averaging this quantity over time then allows to write

$$\langle \vec{F}(t)\rangle = \langle (\vec{p}\cdot\nabla)\vec{E}(t)\rangle = \langle (\alpha\vec{E}(t)\cdot\nabla)\vec{E}(t)\rangle = \frac{\alpha}{2}\nabla\langle E(t)^2\rangle = \frac{\alpha}{2}\nabla\frac{E_{\mathrm{m}}^2}{2} = \frac{\alpha c\mu_0}{2}\partial_x I(x).$$

The standard differential equation for SHO can be retrieved from this as

$$m\ddot{x} = -\frac{2\alpha c\mu_0 I_0}{W_0^2}x = -\kappa_x x \implies \boxed{\Omega = \sqrt{\frac{2\alpha c\mu_0 I_0}{W_0^2}}}.$$

Here, κ_x is collectively known as the trap stiffness of the OT in the x-direction. To find the force due to radiation in the z-direction (as photons are directed on this axis), one can use the formula given in the problem for the power of radiation due to a rotating dipole[6]. As $P = \vec{F}\cdot\vec{v}$, one can write that $F = \frac{P}{c}$. Thus,

$$F = \frac{\mu_0\alpha^2E^2\omega^4}{12\pi c^2} = \frac{\mu_0^2\alpha^2\left(\frac{2\pi c}{\lambda}\right)^4}{6\pi c}I(z) = \frac{8\pi^3c^3\mu_0^2\alpha^2}{3\lambda^4}I_0\left(1-\frac{2z^2}{W_0^2}\right).$$

This must be equal to the gradient force under equilibrium, thus creating a quadratic equation on x:

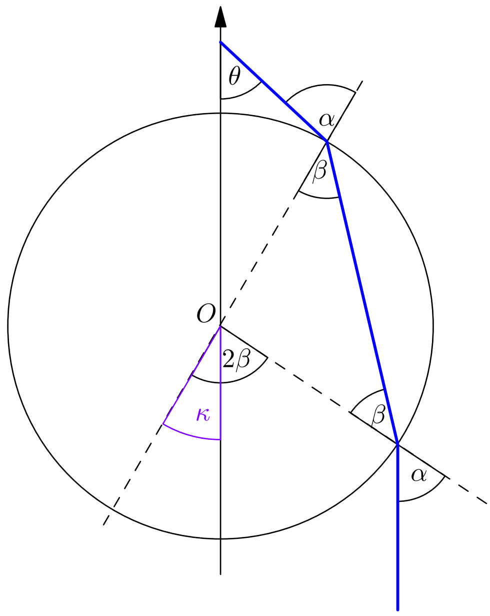
$$F_{\mathrm{rad}} = F_{\mathrm{grad}} \implies \frac{8\pi^3c^3\mu_0^2\alpha^2}{3\lambda^4}I_0\left(1-\frac{2z^2}{W_0^2}\right) = -\frac{2\alpha c\mu_0 I_0}{W_0^2}z.$$

Substituting $\zeta \equiv \frac{4\pi^3c^2\mu_0\alpha W_0^2}{3\lambda^4}$ yields a quadratic of

$$\frac{2\zeta}{W_0^2}z^2 - z - \zeta = 0 \implies \boxed{z = \frac{W_0^2}{4\zeta}\left(1-\sqrt{1-\frac{8\zeta^2}{W_0^2}}\right)}.$$

[6] This is well derived from the Poynting vector.

(b) In the Mie spectrum, $\lambda \gg R$ which implies that the intensity and electric field distribution over the sphere is no longer homogenous. Create a ray diagram:



The deviation in angle can be described by the angle κ or

$$\kappa = 2(\alpha - \beta) = 2\left(\alpha - \arcsin\left(\frac{m}{n}\sin\alpha\right)\right).$$

The change in momentum of photons that move towards the sphere can then be written as

$$\Delta p = \frac{E}{c}(1-\cos\kappa) \implies \mathrm{d}F = \frac{\Delta p}{\Delta t} = 2I\sin^2\left(\alpha - \arcsin\left(\frac{m}{n}\sin\alpha\right)\right)\mathrm{d}A.$$

We can use the law of cosines for a point away from the origin to find ρ:

$$\rho^2 = x^2 + (R\sin\theta)^2 - 2xR\sin\theta\cos.$$

Thus, putting everything together:

$$F_z = \frac{2I_0R^2}{c}\int_0^{\pi/2}\int_0^{2\pi}\sin^2\left(\alpha - \arcsin\left(\frac{m}{n}\sin\alpha\right)\right)\exp\left(\frac{-2\rho^2}{W_0^2}\right)\sin\theta\cos\theta\mathrm{d}\theta\mathrm{d}.$$

Similarly, you can write the forces in the $\hat{x}$ direction which will be similar to F_z:

$$F_x = \frac{I_0R^2}{c}\int_0^{\pi/2}\int_0^{2\pi}\sin\left(\alpha - \arcsin\left(\frac{m}{n}\sin\alpha\right)\right)\exp\left(\frac{-2\rho^2}{W_0^2}\right)\sin\theta\cos\theta\cos\mathrm{d}\theta\mathrm{d}.$$

The net force is thus $F = \sqrt{F_x^2 + F_z^2}$. The torque acting on the sphere can be written as a volume integral over the cross product of distance and force or:

$$\tau = \int_V \vec{r} \times \vec{F}.$$

Here $\vec{r}$ will have different lengths along each direction, or $\vec{r} = (R\sin\alpha\cos, R\sin\alpha\sin, R\cos\alpha)$.

(c) The nanosphere has an angular momentum $L = I\omega$ which implies the torque acting on it to be

$$\tau = \frac{\mathrm{d}L}{\mathrm{d}t} = I\dot{\omega} = \kappa\omega.$$

We now have a simple first order differential equation whose solution is

$$\omega' = \omega_0\exp\left(\frac{5\kappa}{2mR^2}t\right).$$

As the nanosphere is an induced dipole, will it now lose power due to radiative losses? The answer turns out to be no, as the nanosphere rotates on an axis that is perpendicular to the direction of its dipole moment. With no dipole moment directed along the axis of rotation, no radiative power will be lost.

Grading Scheme

A1	It's stated (or taken under consideration) that the entire field on the microsphere is uniform and explanation how oscillations occur is presented	0.50
A3	Intensity and electric field are related under Poynting's vector	1.00
A4	Coefficient α of dipole moment p is calculated	1.00
A5	Gradient force averaged over time is derived	1.00
A6	Answer for Ω	0.75
A7	Radiation force F_{rad} is found and expressed with λ	1.00
A8	Answer for equilibrium position x	0.75
B1	Correct ray diagram and proper deviation angle is derived	2.00
B2	Finds a proper expression for infintessimal force due to momentum change	1.00
B3	Uses law of cosines to obtain a general expression for ρ	0.50
B4	Integral expression for force in x, y, z directions is correct	0.75
B5	Integral expressions for torque is correct	0.75
C1	Differential equation is created and solved	1.0
C2	Correct answer is expressed (can be in a piecewise function)	0.50
C3	Accounts for forces due to radiation and is able to show it does not affect rotation	0.50

Acoustic Levitation

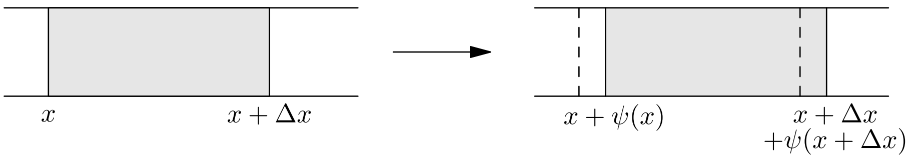
(d) For either case, there are different processes happening which require different analysis.

When f is low

By the condition $f \ll c_s/A$, the acoustic wave produced by the piston travels much faster than the piston as $c_s \gg fA$. For a general traveling wave, we define:

$$\psi(x,t) = A\sin(kx - 2\pi ft)$$

where ψ refers to the longitudinal displacement of the wave and k is the wavenumber. Now consider a part of the gas that is in equilibrium followed by the gas in a later state.



We can approximate $\psi(x+\Delta x)$ to be $\psi(x)+\Delta\psi$. Note that the volume of this slice of gas can be written as $S\Delta x$ but the change in volume of the gas between both successive pictures can be written as $\Delta V = S\Delta\psi$. Furthermore, the ideal gas equation holds for small amplitudes which means that $pV^\gamma = \text{const}$, and by logarithmic differentiation,

$$\frac{\Delta p}{p_0} + \gamma\frac{\Delta V}{V_0} = 0.$$

Thus, we can rewrite the volume terms as:

$$\frac{\Delta V}{V_0} = \frac{S\Delta\psi}{S\Delta x} \to \frac{\partial\psi}{\partial x}.$$

The change in pressure of a segment of the gas at $x=0$ can be characterized as

$$\Delta p(0,t) = -\gamma p_0 \left.\frac{\partial\psi}{\partial x}\right|_{x=0} = \gamma p_0 Ak\cos(2\pi ft).$$

Now the force acting on the piston due to this change in pressure can be accurately described as $\Delta F = S\Delta p(0,t)$. Power can be written as $P = \vec{F}\cdot\vec{v}$ and we know that since $\psi = A\cos(2\pi ft)$, then $v = -2\pi Af\sin(2\pi ft)$. The wavenumber is written as $k = \frac{2\pi}{\lambda} = \frac{2\pi f}{c_s}$. The total and average power acting on the piston for each quarter cycle is then,

$$P = 4\pi^2\gamma p_0 A^2 f^2 c_s\cos^2(2\pi ft) \implies \langle P\rangle = 2\pi^2\gamma f^2 p_0 A^2 c_s.$$

When f is very high

In the limit of $f \gg c_s/A$, the piston will start to move much faster than the actual molecules in the air. Therefore, at some point in time, the piston will actually not have "contact" with the molecular layer. This is because after pushing the gas molecules, there will be an empty void which will eventually be replaced at a distance $d \sim c_s/f$. The piston will only exert a force (and hence power) upon having contact with the layer of gas. Suppose the piston moves a distance $A-d$ in a time τ. We can then find that

$$A\cos(2\pi f\tau) = A - \frac{c_s}{f} \implies 2A\sin^2(\pi f\tau) = \frac{c_s}{f}.$$

Assuming τ is small, we can use approximations such that

$$2\pi^2 Af^2\tau^2 = \frac{c_s}{f} \implies \tau = \frac{1}{\pi}\sqrt{\frac{c_s}{2Af^3}}.$$

The entire cycle takes a time $t = 1/f$ which means we must consider the interval $(\tau, 1/f)$. We can use a similar process as in the analysis for $f \ll c_s/A$ now. Except this time, to find the power used, we will find the energy per cycle and divide by $1/f$. The potential energy per unit length can be written as

$$\mathrm{d}U = \frac{1}{2} S\gamma p_0 \left(\frac{\partial \psi}{\partial x}\right)^2 \implies \Delta U = \frac{1}{2} S\gamma p_0 \int_{\tau}^{1/f} \left(\frac{\partial \psi}{\partial t}\right)^2 \mathrm{d}t.$$

Thus, we must now evaluate:

$$\Delta U = 8S\gamma p_0 \pi^4 A^2 f^4 \cos^2\left(2\pi f\left(\frac{1}{f} - \tau\right)\right).$$

Having all constants be defined under ξ, we can now write

$$\Delta U \approx \xi\left(1 - \left(2\pi f\left(\frac{1}{f} - \frac{1}{\pi}\sqrt{\frac{c_s}{2Af^3}}\right)\right)^2\right) = \xi\left(1 - 4\pi^2 f^2\left(\frac{1}{f^2} - \frac{2}{\pi}\sqrt{\frac{c_s}{2Af^5}} + \frac{1}{\pi^2}\frac{c_s}{2Af^3}\right)\right).$$

The average power used is thus

$$\langle P\rangle = \frac{\Delta U}{1/f} = 8S\gamma p_0 \pi^3 A^2 f^5\left(\frac{1}{f^2} - \frac{2}{\pi}\sqrt{\frac{c_s}{2Af^5}} + \frac{1}{\pi^2}\frac{c_s}{2Af^3}\right).$$

(e) The internal energy of n moles of the gas is given by $U = nC_V\Delta T$. The power dissipated by the shockwave is then

$$P = \frac{\mathrm{d}U}{\mathrm{d}t} = \frac{\mathrm{d}n}{\mathrm{d}t} C_V \Delta T.$$

Note that $\Delta n = \frac{\rho S\Delta x}{m}$ which means that $\mathrm{d}n/\mathrm{d}t = \rho S c_s/m$ where c_s is the speed of the sound wave. Hence, the power dissipated can be written as $P = 3R\rho S c_s \Delta T/2m$. Thus, the change in temperature is when both power generation and dissipation is equated. Note that the process is adiabatic as the wavelength of the sound waves are much larger than the mean free path of the gas molecules implying slow motion.

$$\frac{3R\rho S c_s \Delta T}{2m} = P_{\text{adb}} \implies \Delta T = \frac{2P_{\text{adb}} m}{3R\rho S c_s}.$$

(f) The force on the cylinder will be

$$F = -\pi R^2(p(y+h) - p(y)) = -\pi R^2 h\frac{\mathrm{d}P}{\mathrm{d}y}.$$

From part D, we know that

$$\Delta p(y,t) = \gamma p_0 Ak\cos(ky - 2\pi ft) \implies \frac{\mathrm{d}p}{\mathrm{d}y} = \gamma p_0 Ak\sin(ky - 2\pi ft).$$

From this, we can see that the cylinder will levitate near the pressure nodes, or where $\frac{\mathrm{d}P}{\mathrm{d}y} = 0$. These will typically be compressions as they imply a stable equilibrium. As the waves are travelling, the puck will float up with the sound wave and then fall repeating the same process over again implying oscillations. For true levitate, we would require the waves to be standing.

Grading Scheme

D1	Correct reasoning for what happens to the piston in the limit of $f \ll c_s/A$	0.25
D2	Shows ideal gas equation holds and relates Δp and ΔV	1.00
D3	Rewrites $\Delta V/V_0$ and derives power $P(t)$	1.25
D4	Correct expression for average power $\langle P \rangle$	0.75
D5	Correct reasoning for what happens to the piston in the limit of $f \gg c_s/A$	0.50
D6	Finds characteristic time τ for when piston hits the "molecular wall"	0.50
D7	Finds potential energy per unit length per cycle	1.25
D8	Correct expression for average power	0.75
E1	Relates internal energy and power dissipation (alternate: uses equipartition theorem)	3.00
E2	Correct answer for ΔT	1.00
F1	Relates force F and pressure gradient p to show sinusoidal nature	1.00
F2	Correct reasoning for how the cylinder levitates and where it does	1.00

T2: Thomas Precession

Sucessive Transformations

In this section, we examine what happens when two successive Lorentz transformations are applied in non-parallel directions.

1. **(6 pts.)** Consider the three reference frames S_1, S_2, and S_3. Events as seen from frame S_i will be labeled with the space-time coordinates (x_i, y_i, z_i, t_i), for $i = 1, 2, 3$. All three frames coincide at $(0, 0, 0, 0)$. Suppose frame S_2 travels with velocity βc in the x_1-direction of frame S_1 and frame S_3 travels with velocity $\beta'_x c\hat{\mathbf{i}}_2 + \beta'_y c\hat{\mathbf{j}}_2$ with respect to frame S_2.

   Perform two successive Lorentz transformations: one expressing the S_3 coordinates in terms of the S_2 coordinates and another expressing the S_2 coordinates in terms of the S_1 coordinates. Then, as the final answer, express the S_3 coordinates in terms of the S_1 coordinates.

   Assume that $\beta' = (\beta'^2_x + \beta'^2_y)^{1/2} \ll \beta$ and work to first order.

2. **(5 pts.)** Now, find the velocity of S_3 in S_1 with the appropriate velocity addition. Perform a single Lorentz transformation to express the S_3 coordinates in terms of the S_1 coordinates. Once again, work to first order. The answer will not be the same as part 1.

3. **(3 pts.)** Show that your answer in Problem 1 differs from your answer in Problem 2 by a spatial rotation. In other words, two successive Lorentz transformations in non-parallel directions cannot be combined as one Lorentz transformation. Rather, they are the combination of one Lorentz transformation and one spatial rotation. Determine the magnitude and direction of this spatial rotation in the current setup.

   Then argue and explain why the answer in Problem 1 and not Problem 2 is the correct transformation.

Precession Frequency

In this section, we examine the precession of the electron's spin magnetic moment within the hydrogen atom.

Electrons possess an attribute known as *spin*. One can imagine the electron as being a small, spherical charged particle that is spinning on some axis. Although this mental picture is not physically correct, it is enough for us. From this mental picture, we can gather that the electron will possess a dipole moment and angular momentum *intrinsic* to itself and not induced by any orbital motion. Respectively, these are known as the *spin magnetic moment* and the *spin angular momentum* that behave, in our model, just as their classical counterparts do. These two vector quantities, $\boldsymbol{\mu}$ and $\mathbf{L}$ respectively, are related by

$$\boldsymbol{\mu} \simeq -\frac{e}{m_e}\mathbf{L},$$

which is determined by experiment.

In this section, we will use the Bohr model of the hydrogen atom, where the electron circles the proton at some radius r, pulled in by the Coulomb force.

4. **(3 pts.)** In the laboratory frame, the electron orbits the proton with some velocity v in the x-y plane. Now switch to the instantaneous rest frame of the electron, where the proton moves with speed v relative to the stationary electron. The moving proton will then induce some magnetic field at the electron's location.

   Suppose the spin angular momentum of the electron points in some direction other than the direction of the magnetic field. Because the electron also possesses a spin magnetic moment, it will precess as a result of the torque done on it. Find the angular frequency of the precession of the electron's spin in terms of r and whatever fundamental constants.

   Ignore any relativistic effects.

5. **(6 pts.)** This problem will use the answer from Part 1. Once again, assume that the electron is orbiting in the x-y plane.

   Consider the instantaneous rest frame of the electron at some time t, S_2. Also consider the instantaneous rest frame of the electron at some time $t + \mathrm{d}t$, S_3. Relative to the laboratory frame, S_1, S_2 will have velocity $\mathbf{v}$. S_3 will have a velocity $\mathrm{d}\mathbf{v}$ relative to S_2, but $\mathbf{v}$ and $\mathrm{d}\mathbf{v}$ won't be parallel.

   The result from Part 1 tells us that S_3 will experience an infinitesimal rotation in this time $\mathrm{d}t$ with respect to the laboratory frame. Since the electron is continuously accelerating, its rest frame, S_3, must then rotate continuously relative to the lab frame. Assume that the spin of the electron always points in the same direction in its rest frame. If the spin is not pointing in the z direction, find the angular frequency of the spin's precession in terms of r and whatever fundamental constants. Ignore the effects of the previous problem.

   Note: although relativistic effects cannot be ignored, you can assume that the electron's velocity v is not comparable to the speed of light in the calculation.

6. **(2 pts.)** Combine your answers from parts 4 and 5 and find the relativistically correct angular frequency of the precession of the electron's spin.

This problem deals with a phenomenon known as *Thomas precession*. It is named after Llewellyn Thomas. Thomas did not discover this phenomenon, but he was the first to discover that its application was important for the electron in the hydrogen atom. He did so in one-page letter, which was printed in *Nature* in 1926. Thomas's contribution was crucial for the calculation of the spin-orbit interaction in the hydrogen atom, which successfully explained the fine structure of the hydrogen spectral lines.

This entire problem is essentially an explanation of his contribution and can all be found in his one-page letter to the editor.

Part 1

Problem 1

(x_2, y_2, z_2, t_2) is related to (x_1, y_1, z_1, t_1) by a simple Lorentz transformation:

$$ct_2 = \gamma(ct_1 - \beta x_1) \tag{1a}$$
$$x_2 = \gamma(x_1 - \beta ct_1) \tag{1b}$$
$$y_2 = y_1 \tag{1c}$$
$$z_2 = z_1, \tag{1d}$$

where γ is the Lorentz factor correlated with β. To go from S_2 to S_3, we must Lorentz transform in a direction that's not the x axis. Our strategy will be to rotate S_2 to S_2' such that S_3 travels along the x_2' axis. Then we can Lorentz transform to find S_3' and rotate back to S_3.

First,

$$ct_2' = ct_2 \tag{2a}$$
$$x_2' = \cos\theta x_2 + \sin\theta y_2 = \frac{\beta_x'}{\beta'}x_2 + \frac{\beta_y'}{\beta'}y_2 \tag{2b}$$
$$y_2' = -\sin\theta x_2 + \cos\theta y_2 = -\frac{\beta_y'}{\beta'}x_2 + \frac{\beta_x'}{\beta'}y_2 \tag{2c}$$
$$z_2' = z_2. \tag{2d}$$

If the Lorentz factor for β' is γ', we can write

$$ct_3' = \gamma'(ct_2' - \beta' x_2') \tag{3a}$$
$$x_3' = \gamma'(x_2' - \beta' ct_2') \tag{3b}$$
$$y_3' = y_2' \tag{3c}$$
$$z_3' = z_2'. \tag{3d}$$

Then we can rotate back and find

$$ct_3 = ct_3' \tag{4a}$$
$$x_3 = \frac{\beta_x'}{\beta'}x_3' - \frac{\beta_y'}{\beta'}y_3' \tag{4b}$$
$$y_3 = \frac{\beta_y'}{\beta'}x_3' + \frac{\beta_x'}{\beta'}y_3' \tag{4c}$$
$$z_3 = z_3'. \tag{4d}$$

Combining all of (1), (2), (3), and (4), we can write

$$
\begin{aligned}
ct_3 &= \gamma\gamma' ct_1 - \gamma\gamma'\beta x_1 - \gamma\gamma'\beta'_x x_1 + \gamma\gamma'\beta'_x\beta ct_1 - \gamma'\beta'_y y_1 \\
&\simeq \gamma(1+\beta'_x\beta)ct_1 - \gamma(\beta+\beta'_x)x_1 - \beta'_y y_1 && \text{(5a)} \\
x_3 &= \gamma\gamma'\left(\beta\beta'_x + \frac{\beta'^2_x}{\beta'^2}\right)x_1 - \gamma\gamma'\left(\beta'_x + \frac{\beta\beta'^2_x}{\beta'^2}\right)ct_1 + \frac{\gamma'\beta'_x\beta'_y}{\beta'^2}y_1 \\
&\quad + \frac{\gamma\beta'^2_y}{\beta'^2}x_1 - \frac{\gamma\beta\beta'^2_y}{\beta'^2}ct_1 - \frac{\beta'_x\beta'_y}{\beta'^2}y_1 \\
&\simeq -\gamma(\beta+\beta'_x)ct_1 + \gamma(1+\beta\beta'_x)x_1 && \text{(5b)} \\
y_3 &= \gamma\gamma'\left(\beta\beta'_y + \frac{\beta'_x\beta'_y}{\beta'^2}\right)x_1 - \gamma\gamma'\left(\beta'_y + \frac{\beta'_x\beta'_y\beta}{\beta'^2}\right)ct_1 + \frac{\gamma'\beta'^2_y}{\beta'^2}y_1 \\
&\quad - \frac{\gamma\beta'_x\beta'_y}{\beta'^2}x_1 + \frac{\gamma\beta\beta'_x\beta'_y}{\beta'^2}ct_1 + \frac{\beta'^2_x}{\beta'^2}y_1 \\
&\simeq -\gamma\beta'_y ct_1 + \gamma\beta\beta'_y x_1 + y_1 && \text{(5c)} \\
z_3 &= z_1. && \text{(5d)}
\end{aligned}
$$

Here, everything has been done to first order, which includes writing $\gamma' \simeq 1$.

Problem 2

To transform from S_1 directly to S_3, we need the proper velocity addition. Let the velocity of S_3 in S_1 be defined by the components β''_x and β''_y. Then

$$\beta''_x = \frac{\beta+\beta'_x}{1+\beta\beta'_x} \text{ and } \beta''_y = \frac{\beta'_y}{\gamma(1+\beta\beta'_x)}. \tag{6}$$

We further define $\beta'' = (\beta''^2_x + \beta''^2_y)^{1/2}$ and γ'' as the Lorentz factor corresponding to β''.

Note that the following approximations are true to first order:

$$
\begin{aligned}
\beta''_x &\simeq \beta + \frac{\beta'_x}{\gamma^2} && \text{(7a)} \\
\gamma'' &\simeq \gamma(1+\beta\beta'_x) && \text{(7b)} \\
\beta'' &\simeq \beta''_x. && \text{(7c)}
\end{aligned}
$$

Once again, the Lorentz transformation needs to be made in a direction other than the x direction. We employ a similar scheme. The details of the calculation are not shown here, since they are similar to what has been done previously. The final result is

$$
\begin{aligned}
ct'_3 &= \gamma'' ct_1 - \gamma''\beta''_x x_1 - \gamma''\beta''_y y_1 \\
&\simeq \gamma(1+\beta\beta'_x)ct_1 - \gamma(\beta+\beta'_x)x_1 - \gamma\beta''_y y_1 && \text{(8a)} \\
x'_3 &= \frac{\gamma''\beta''^2_x}{\beta''^2}x_1 + \frac{\gamma''\beta''_x\beta''_y}{\beta''^2}y_1 - \gamma''\beta'' ct_1 + \frac{\beta''^2_y}{\beta''^2}x_1 - \frac{\beta''_x\beta''_y}{\beta''^2}y_1 && \text{(8b)} \\
&\simeq -\gamma(\beta+\beta'_x)ct_1 + \gamma(1+\beta\beta'_x)x_1 + (\gamma-1)\frac{\beta''_y}{\beta}y_1 && \text{(8c)} \\
y'_3 &= \frac{\gamma''\beta''_x\beta''_y}{\beta''^2}x_1 + \frac{\gamma''\beta''^2_y}{\beta''^2}y_1 - \gamma''\beta''_y ct_1 - \frac{\beta''_x\beta''_y}{\beta''^2}x_1 + \frac{\beta''^2_x}{\beta''^2}y_1 \\
&\simeq -\gamma\beta''_y ct_1 + (\gamma-1)\frac{\beta''_y}{\beta}x_1 + y_1 && \text{(8d)} \\
z'_3 &= z_1. && \text{(8e)}
\end{aligned}
$$

Note that all terms β''^2_y/β''^2 are second order.

Problem 3

Let us write S_3 in terms of S'_3. Using (6), we can write (5) in terms of β''_y instead of β'_y. As we will see in Problem 5, β''_y, the laboratory-observed difference in velocity between S_2 and S_3, is more physically relevant. This is

$$ct_3 \simeq \gamma(1+\beta'_x\beta)ct_1 - \gamma(\beta+\beta'_x)x_1 - \gamma\beta''_y y_1 \tag{9a}$$
$$x_3 \simeq -\gamma(\beta+\beta'_x)ct_1 + \gamma(1+\beta\beta'_x)x_1 \tag{9b}$$
$$y_3 \simeq -\gamma^2\beta''_y ct_1 + \gamma^2\beta\beta''_y x_1 + y_1 \tag{9c}$$
$$z_3 = z_1 \tag{9d}$$

We can see that this is different from (8). Let us invert (8) such that we have (x_1, y_1, z_1, t_1) in terms of (x'_3, y'_3, z'_3, t'_3). This is just the same Lorentz transformation in Problem 2, except with the velocities reversed. So

$$ct_1 \simeq \gamma(1+\beta\beta'_x)ct'_3 + \gamma(\beta+\beta'_x)x'_3 + \gamma\beta''_y y'_3 \tag{10a}$$
$$x_1 \simeq \gamma(\beta+\beta'_x)ct'_3 + \gamma(1+\beta\beta'_x)x'_3 + (\gamma-1)\frac{\beta''_y}{\beta}y'_3 \tag{10b}$$
$$y_1 \simeq \gamma\beta''_y ct'_3 + (\gamma-1)\frac{\beta''_y}{\beta}x'_3 + y'_3 \tag{10c}$$
$$z_1 = z'_3. \tag{10d}$$

Plugging (10) into (9), we find

$$ct_3 \simeq ct'_3 \tag{11a}$$
$$x_3 \simeq x'_3 - (\gamma-1)\frac{\beta''_y}{\beta}y'_3 \tag{11b}$$
$$y_3 \simeq (\gamma-1)\frac{\beta''_y}{\beta}x'_3 + y'_3 \tag{11c}$$
$$z_3 = z'_3. \tag{11d}$$

This is the equation for an infinitesimal rotation (keeping in mind that $\beta''_y/\beta \ll 1$). Thus, to go from S'_3 to S_3, we must rotate about the $+z$ axis an angle

$$-(\gamma-1)\frac{\beta''_y}{\beta}.$$

To reiterate, β''_y is the relative velocity in the y direction between frames S_3 and S_2, as viewed from the lab frame S_1.

We expect (x_3, y_3, z_3, t_3) to be the correct form. The Lorentz transformation is the most general tool. The velocity addition used in problem 2 is dependent on the Lorentz transformation.

Part 2

Problem 4

We are given

$$\vec{\mu} = -\frac{e}{m_e}\vec{L}. \tag{12}$$

Suppose the electron is orbiting counter-clockwise in the x-y plane. In the instantaneous rest frame of the electron, the proton travels with speed v a distance r away from the electron. By the law of Biot and Savart, the magnetic field at the location of the electron is

$$B = \frac{\mu_0 ev}{4\pi r^2}, \tag{13}$$

which points in the positive z direction with our choice of coordinates.

Since we are using the Bohr model, the acceleration of the electron is given by the equation

$$\frac{v^2}{r} = \frac{e^2}{4\pi\epsilon_0 r^2 m_e} \tag{14}$$

which gives

$$v = \left(\frac{e^2}{4\pi\epsilon_0 r m_e}\right)^{1/2}. \tag{15}$$

Now the electron will experience a torque as a result of this magnetic field because it possesses a spin magnetic moment. This torque is

$$\vec{\tau} = \vec{\mu} \times \vec{B} = -\frac{e}{m_e}\vec{L} \times \vec{B}. \tag{16}$$

Therefore,

$$\left|\frac{d\vec{L}}{dt}\right| = \frac{eLB\sin\theta}{m_e}. \tag{17}$$

θ is the angle between $\vec{L}$ and $\vec{B}$. Thinking in analogy to $v = r\omega$, we find that the angular velocity of the precession of $\vec{L}$ is

$$\omega_L = \frac{eB}{m}, \tag{18}$$

which will be in the positive z direction. (We have used the subscript L because this precession is often called the *Larmor precession*.) Applying (13) and (15), we find that

$$\omega_L = \frac{\mu_0 e^3}{4\pi m_e r^2}\left(\frac{1}{4\pi\epsilon_0 m_e r}\right)^{1/2}. \tag{19}$$

Problem 5

Once again, assume the electron is orbiting counter-clockwise in the x-y plane.

In analogy to Problem 1, S_1 is the lab frame. S_2 is the rest frame of the electron when it is travelling with velocity $\vec{v}$ at time t. S_3 is the rest frame of the electron when it is travelling with velocity $\vec{v} + \vec{a}dt$ at time $t + dt$. To keep the analogy, let $\vec{v}$ be in the x direction of S_1. Then $\vec{a}dt$ is in the y direction of S_1 and S_2. (Since the electron is orbiting counter-clockwise, it must also be the positive y direction.)

First, $\beta = v/c$ as defined in Problem 1. Second, note that $\vec{a}dt$ is the relative velocity between S_3 and S_2 as measured in the lab frame S_1. Therefore, $\beta_y'' = a\,dt/c$. Combining these facts, the infinitesimal rotation angle between S_2 and S_3 will be

$$d\theta = -(\gamma - 1)\frac{a\,dt}{v}, \tag{20}$$

in the positive z direction. Since we can also assume that $v/c \ll 1$, $\gamma - 1$ simplifies to $v^2/2c^2$. We can continue to repeat this process as the electron goes around in a circle. Therefore, this continuous infinitesimal rotational angle will manifest as an angular velocity: the angular frequency with which the frame S_3 rotates relative to the lab frame. This angular frequency is

$$\omega_T = -\frac{av}{2c^2} = -\frac{v^3}{2c^2 r} \tag{21}$$

in the positive z direction. (We have used the subscript T because this frequency is called the Thomas precession frequency.)

Using (15), (21) can be rewritten as

$$\omega_T = -\frac{e^3}{8\pi\epsilon_0 c^2 m_e r^2}\left(\frac{1}{4\pi\epsilon_0 m_e r}\right)^{1/2}. \tag{22}$$

Problem 6

The final precession frequency of the electron will be $\omega_L + \omega_t$. Note that $c = (1/\mu_0\epsilon_0)^{1/2}$. Thus, (22) can be rewritten as

$$\omega_T = -\frac{\mu_0 e^3}{8\pi m_e r^2}\left(\frac{1}{4\pi\epsilon_0 m_e r}\right)^{1/2}, \tag{23}$$

which is exactly half the result of Problem 4. Therefore, the precession frequency of the electron is

$$\omega = \frac{\mu_0 e^3}{8\pi m_e r^2}\left(\frac{1}{4\pi\epsilon_0 m_e r}\right)^{1/2}. \tag{24}$$

This relativistic correction of 1/2 because known as the *Thomas half* and finally matched the experimental determination of hydrogen's fine structure with the quantum mechanical prediction. Just a few year's later, Dirac would propose his Dirac Equation, which "came installed" with relativistic mechanics, thus eliminating the need for these kinds of calculations.

Grading Scheme

P1	For correct Lorentz transformation from S_1 to S_2	1 pt
P1	For correct Lorentz transformation from S_2 to S_3	2 pt
P1	For correct Lorentz transformation from S_1 to S_3	3 pt
P1	For not expressing the answer to first order with the given approximation. Half credit for algebraic mistakes	-2 pt
P2	For the correct velocity.	2 pt
P2	For the correct Lorentz transformation.	4 pt
P2	For not expressing to first order. Half credit for algebra mistakes.	-2 pt
P3	For correct magnitude.	2 pt
P3	For correct direction.	1 pt
P4	For the correct magnetic field at the electron's location within its rest frame.	0.5 pt
P4	For the correct torque applied on the electron due to its spin magnetic moment.	0.5 pt
P4	For the correct precession frequency. Half credit for algebra mistakes.	2 pt
P5	For the correct analogy (assigning values for β, β'_x, β'_y, etc) to parts 1, 2, and 3.	2 pt
P5	For the correct answer. Half credit for algebra mistakes.	4 pt
P5	For not assuming $v \ll c$.	-1 pt
P6	For the correct answer.	2 pt
P6	For not making the substitution $c = (\mu_0\varepsilon_0)^{-1/2}$.	-1 pt

T3: Moving Media

Interesting phenomena can arise in situations where there are 2 media that are moving with respect to each other. In particular, objects can move at much faster speeds than the relative speeds of the media, without using any energy. In this problem, we explore two such examples of this effect.

Moving Cylinders

Suppose we have three cylinders, two small cylinders and a large cylinder, of radii r and R. The frictionless pivots (centers) of the cylinders are attached to a massless triangular frame, such that the large cylinder is in contact with the two small cylinders but the two small cylinders are not touching each other. The small cylinders each have a thin groove along their circumferences (which does not affect the moment of inertia significantly), so that the large cylinder makes contact with the small cylinder at a point with radial distance αr from the center of the small cylinder. The axes of all cylinders are perpendicular to the plane of the triangular frame. The system is placed on a level ground and a long flat horizontal board is put on top of the large cylinder, with the two small cylinders touching the ground (making contact at their outer edge with radial distance r, not αr). Assume that the friction due to contact between all surfaces is large enough to prevent any slipping.

1. **(4 pts.)** The board is moved with speed v in a direction perpendicular to the axes of the cylinders. Find the speed of the cylinder system.

2. **(5 pts.)** The mass of the small and large cylinders are m and M, respectively. The mass of the board is m'. If at a moment in time the board is pushed with speed v and acceleration a, find the power P required to push the board. Assume the cylinders have uniform mass distribution.

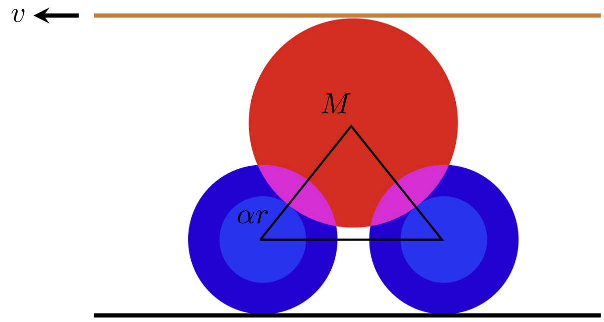


Figure 4: A visual of the three cylinder setup.

Windsurfing

In windsurfing, it is possible to sail faster than the wind without using any energy. Suppose we have a sailboat moving on a large, motionless body of water. The air of density ρ is moving at a speed v uniformly in one direction. If the sailboat is pointed in a certain direction and moves in that direction with velocity $\mathbf{u}$, the drag force from the water $\mathbf{F}$ satisfies $\mathbf{F} \cdot \mathbf{u} = -\gamma u^2$, where γ can be assumed to be a constant drag coefficient.

3. **(10 pts.)** If the wind is moving in the $\hat{x}$ direction, what is the maximum possible sustainable x-component of velocity for the sailboat? Assume that the sailboat can neither generate nor store energy in its interaction with the air. Also, the effective cross-sectional area of the sail is A (this is the component of cross-sectional area that is perpendicular to the wind in the reference frame of the sailboat).

4. **(3 pts.)** What is the power dissipated due to the interaction with the water?

5. **(3 pts.)** It seems that the law of conservation of energy is being violated, as the speed of the sailboat isn't changing despite heat generation in the water. Explain why energy is still conserved.

Moving Cylinders

(a) Consider the reference frame of the cylinder system. Since friction is large enough to prevent any slipping, if the speed of the edge of the large cylinder is u, then the speed of the edge of the small cylinder is $\frac{u}{\alpha}$. Thus, the difference in speeds $u(\frac{1}{\alpha}-1)=v$. The speed of the cylinder system is $\frac{u}{\alpha}=\frac{v}{1-\alpha}$.

(b) The power P is equal to the rate of change of kinetic energy of the system. The board increases kinetic energy at a rate $m'va$, the cylinders increase rotational kinetic energy at rates $I\omega\dot{\omega}$. The rate of increase of translational kinetic energy of the cylinder system is $\frac{M+2m}{(1-\alpha)^2}va$. Adding up all the contributions, one gets

$$\begin{aligned} P &= m'va + \frac{M+2m}{(1-\alpha)^2}va + \frac{1}{2}MR^2\left(\frac{\alpha}{(1-\alpha)R}\right)^2 va + 2\cdot\frac{1}{2}mr^2\left(\frac{1}{(1-\alpha)r}\right)^2 va \\ &= \left(m' + \frac{M+3m}{(1-\alpha)^2} + \frac{1}{2}\frac{M\alpha^2}{(1-\alpha)^2}\right)va. \end{aligned}$$

Windsurfing

(c) Suppose the sailboat is pointed at some angle from the direction of the wind. We work in the reference frame of the sailboat throughout this solution, and consider the system containing the air and the sailboat. The water does no work on this system, because it is only in contact with the sailboat, which is not moving in this frame. To achieve the optimal state, we want no energy to be lost (except the heat in the water). Thus, the only thing that the sailboat can do is change the direction of the velocity vector of the wind it intercepts, leaving the magnitude the same! Suppose the wind is coming in at a speed x, and the water is moving at a speed y (all speeds are relative to the sailboat). Let θ be the angle between these two velocities. Then by law of cosines, we get $\cos\alpha = \frac{x^2+y^2-v^2}{2xy}$. For the sailboat to get the maximum push from the air, the air must be thrown back directly parallel to y. Thus, the force is the rate of momentum transfer, which is $\rho Ax^2(1-\cos\alpha)=\gamma y$. Simplifying, we get $\frac{\rho A}{\gamma}x(2xy-x^2-y^2+v^2)=2y^2$. Taking differentials, we get $\frac{\rho A}{\gamma}((2x^2-2xy)dy+(4xy-3x^2-y^2+v^2)dx)=4ydy$. Also, by geometric relations, we get that the x-component speed of the boat relative to the water in excess of the wind speed (v) is $\frac{y^2-x^2-v^2}{2v}$. Thus, this is maximized when its derivative is zero, which is when $xdx=ydy$. Combining this with the previous differential, we get

$$\frac{\rho A}{\gamma}(2x^3-5x^2y+4xy^2-y^3+v^2y)=4xy$$

This gives us the following two equations and two variables (x and y):

$$(2x^2y-x^3-xy^2+xv^2)=2*\frac{\gamma}{\rho Av}y^2v$$

$$2x^3-5x^2y+4xy^2-y^3+v^2y=4*\frac{\gamma}{\rho Av}xyv$$

Using the numerical value $\frac{\gamma}{\rho Av}=0.05$, we can graph the two equations on a graphing calculator (such as Desmos) and we obtain the solution $(x,y)=(6.17,6.69)v$ that maximizes y^2-x^2. Thus, the maximum x-component of the velocity of the sailboat is approximately $3.9v$.

(d) The power dissipated is given by $F\cdot v=\gamma y^2$. This is approximately $44.8\gamma v^2$.

(e) Energy conservation is not violated, because in the frame of the water, the air had a lot of kinetic energy to begin with, and during the process, the sailboat reduced the speed of part of the air, thus taking energy from the air.

Grading Scheme

A1	Correct ideas when finding the speed of smaller and larger cylinder	2.0
A2	Correct answer for the entire speed of the cylinder system	2.0
B1	Uses the fact that $\Delta E/\Delta t = P$	1.0
B2	Considers the increase in energy of E_{board}	1.0
B3	Considers the increase in energy of $E_{\text{translational}}$	1.0
B4	Considers the increase in energy of $E_{\text{rotational}}$	1.0
B5	Correct final answer	1.0
C1	Correct idea that the energy of the sail comes from wind	1.0
C2	Correct idea that wind velocity's direction can change, but not magnitude	4.0
C3	Expression for maximum force due to wind	2.0
C4	Expression for drag force from water	1.0
C5	Correct answer within 2% (1 pt removed if final equations are correct but not answer and vice versa)	2.0
D1	Correct formula for power ($F \cdot v = \gamma y^2$)	2.0
D2	Correct answer (within 4%)	1.0
E1	Correct explanation that energy comes from wind	3.0

T4: Missing Energy

Part A

1. **(3 pts.)** Consider a simple circuit with two parallel-plate capacitors of capacitance C_1 and C_2 connected to each other using purely conducting wires and a switch. One of the capacitors is initially charged to a voltage V_0, while the other one is completely uncharged. The circuit is kept in a square shaped figure of side length ℓ throughout the problem, while the diameter of the conducting wires is D. Find the initial total energy of the circuit when the switch is open, given by E_0, and a sufficiently long time after the switch is closed, given by E_∞. Calculate the remaining energy $E_\Delta = E_0 - E_\infty$. What is E_Δ for the case $C_2 \to \infty$?

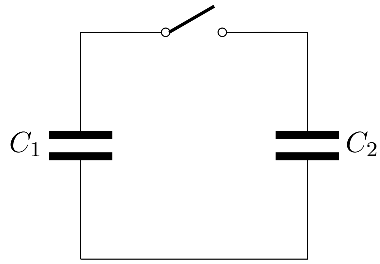


Figure 5: The two parallel plate capacitor-switch circuit.

It seems odd for there to be a difference in energy as the circuit is a closed system. Three young scientists Fermi, Jackson, and Feynman have created different theories to find and verify the correct source of this missing energy.

Thermal Losses: Fermi

To investigate the cause of this missing energy, Fermi assumes that there must be an ohmic resistive load r and a self-inductance L in the circuit responsible for E_Δ.

2. **(3 pts.)** Find the current in the circuit $I(t)$ as a function of time and E_Δ for the circuit.

3. **(1 pt.)** For small values of r, find the oscillation frequency Ω of $I(t)$.

4. **(1 pt.)** For $L = 0$, can Fermi's reasoning be correct for any value of r? If yes, what is this value of r?

Dipole Radiation Losses: Jackson

Jackson believes that the missing energy is dissipated in the form of dipole radiation losses due to the charges accelerating. He assumes that the electric dipole moment of the system remains constant during the process, but the magnetic moment is allowed to vary. Thus, he seeks to determine the maximal possible radiation losses. For this he uses Larmor's formula, which states that for small velocities relative to the speed of light c, total power radiated which the radiation power is defined as:

$$P_r = \frac{2}{3} \cdot \frac{\ddot{m}^2}{4\pi\epsilon_0 c^5}$$

where m is the magnetic moment of the circuit as a function of time. Ignore all relativistic effects and the possible charge accumulation in the wires compared to that on the capacitor plates. Moreover, note that he does not assume any resistance or self-inductance in the circuit in his model.

5. **(4 pts.)** Find the total energy dissipation E_r due to this radiation.

6. **(1 pt.)** For what value of time interval $\Delta\tau$ taken by the charges to move from one capacitor to another, can Jackson's theory be reasoned true?

Kinetic Energy: Feynman

Feynman has the following hypothesis:

> The missing energy goes into the kinetic energy of the charge carriers going from C_1 to C_2. Assume that the mean free path of collisions of the carriers is $\lambda > 2\ell$.

7. **(4 pts.)** Find the total kinetic energy ΔK gained by the carriers during a total charge transfer from C_1 to C_2.

8. **(1 pt.)** Could this be a valid hypothesis to explain the cause of the missing energy? When the charges get completely deposited on the plates of C_2, what happens to this kinetic energy?

Part B

Instead of charging one capacitor using the other, we take an ideal parallel-plate capacitor such that the surface charge density $\pm\sigma$ on its plates is uniform throughout both plates, and that the charges are 'fixed' to the surface as the plate expands. The dimensions of the plates are a, b and the plate separation distance is d. The plate is now stretched quasi-statically by a factor of κ in one of the dimensions such that the dimensions of the capacitor plates are now $\kappa a, b$ but the plate separation remains d.

9. **(6 pts.)** Calculate the work $\mathrm{d}W$ done during stretching the plates of this capacitor. Also write a simplified form of this expression for $d \ll \kappa a, b$.

10. **(1 pt.)** In this case, there are no resistive loads, and since the process of plate expansion is quasi-static, there is no gain in kinetic energy of the charge carriers. Where does the energy disappear in this case?

Solution

T4: Part A

Problem 1

When the switch is open, there is $q_0 = \pm C_1 V_0$ charge on one of the capacitors, while the other capacitor is uncharged. At this time, we have

$$E_0 = \frac{{q_0}^2}{2C_1} = \frac{1}{2}C_1{V_0}^2$$

A long time after the switch is switched on, the potential across both capacitors must be equal. Moreover, by charge conservation we have

$$q_0 = C_1V_0 = q_\infty = (C_1 + C_2)V_\infty \Rightarrow V_\infty = \frac{C_1V_0}{C_1 + C_2}$$

Hence the energy after a long time is

$$E_\infty = \frac{{q_\infty}^2}{2(C_1 + C_2)} = \frac{1}{2}(C_1 + C_2){V_\infty}^2 = \frac{1}{2}\frac{{C_1}^2}{C_1 + C_2}{V_0}^2$$

This means that

$$E_\Delta = E_0 - E_\infty = \frac{1}{2}\frac{C_1C_2}{C_1 + C_2}{V_0}^2 = \frac{1}{2}C_{\text{eq}}{V_0}^2$$

where C_{eq} is the equivalent capacitance of the circuit. For $C_1 = C_2$, this loss is minimal and equal to $E_0/2$. For $C_2 \to \infty$, the whole E_0 energy is lost. In the following problems, we will investigate why there is a loss of energy and work out different explanations of it.

Problem 2

Let q_1 be the charge on the first capacitor at any time t. By conservation of charge, the charge on the second capacitor must be $q_0 - q$. Apply Kirchoff's Voltage Law on the circuit, we have

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + ir + \frac{q}{C_1} - \frac{q_0 - q}{C_2} = 0$$

and

$$i = -\frac{\mathrm{d}q}{\mathrm{d}t}$$

Differentiating the DE gives us

$$\frac{\mathrm{d}^2 i}{\mathrm{d}t^2} + \frac{r}{L}\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{LC_{\text{eq}}}i = 0$$

To solve this standard second order DE, we write its auxillary quadratic equation in z:

$$z^2 + \frac{r}{L}z + \frac{1}{LC_{\text{eq}}} = 0$$

Upon solving this we get

$$z = -\frac{r}{2L} \pm j\sqrt{\frac{1}{LC_{\text{eq}}} - \frac{r^2}{4L^2}}$$

Substituting back $i = e^{jz}$ we obtain

$$i(t) = e^{-\frac{r}{2L}t}\left(c_1 e^{j\sqrt{\frac{1}{LC_{\text{eq}}} - \frac{r^2}{4L^2}}t} + c_2 e^{-j\sqrt{\frac{1}{LC_{\text{eq}}} - \frac{r^2}{4L^2}}t}\right)$$

Using Euler's identity and imposing the boundary conditions $i(t=0) = 0$, $\frac{\mathrm{d}i(t=0)}{\mathrm{d}t} = \frac{V_0}{L}$ we get the current of the form

$$i(t) = \begin{cases} \frac{V_0}{L\sqrt{\frac{1}{LC_{\text{eq}}} - \frac{r^2}{4L^2}}} e^{-\frac{r}{2L}t} \sin\left(\sqrt{\frac{1}{LC_{\text{eq}}} - \frac{r^2}{4L^2}}t\right) & \text{for } \frac{1}{LC_{\text{eq}}} > \frac{r}{2L} \\ \frac{V_0}{L} t e^{-\sqrt{\frac{1}{LC_{\text{eq}}} - \frac{r^2}{4L^2}}t} & \text{for } \frac{1}{LC_{\text{eq}}} = \frac{r}{2L} \\ \frac{V_0}{jL\sqrt{\frac{1}{LC_{\text{eq}}} - \frac{r^2}{4L^2}}} e^{-\sqrt{\frac{1}{LC_{\text{eq}}} - \frac{r^2}{4L^2}}t} \sinh\left(j\sqrt{\frac{1}{LC_{\text{eq}}} - \frac{r^2}{4L^2}}t\right) & \text{for } \frac{1}{LC_{\text{eq}}} < \frac{r}{2L} \end{cases}$$

for overdamping, critical damping, and underdamping. For small values of r and L, the system is likely to undergo underdamping. Also note that

$$E_\Delta = \int_0^\infty (i(t))^2 r \mathrm{d}t$$

Substituting value of $i(t)$ from any of the cases, we find that

$$E_\Delta = \frac{1}{2}\frac{C_1 C_2}{C_1 + C_2}{V_0}^2$$

as found in the first part. This means that Fermi's explanation may be a valid one, if the experimental values suggest the existence of non-zero r and L.

Problem 3

For small values of r, we must have

$$\frac{1}{LC_{\text{eq}}} > \frac{r}{2L}$$

On a first degree of approximation for $r \to 0^+$, this gives us

$$\Omega = \sqrt{\frac{1}{LC_{\text{eq}}} - \frac{r^2}{4L^2}} \approx \sqrt{\frac{1}{LC_{\text{eq}}}}$$

Problem 4

Fermi's reasoning should hold true for all finite values of $r > 0$ for $L \to 0^+$. But note that for zero self-inductance in the circuit, it behaves as a simple RC-circuit with time constant $\tau = rC_{\text{eq}}$. Since relativistic effects are negligible, we must also simultaneously have $r \gg \frac{\ell}{cC_{\text{eq}}}$.

Problem 5

Due to discrepancy in Larmor's formula during the competition, the following solutions to this problem will be given full credit:

Solution 1:

Magnetic moment of the circuit is $m = iA = i\ell^2$, and thus $\ddot{m} = \ell^2 \frac{\mathrm{d}^2 i}{\mathrm{d}t^2}$. Since the capacitors are arranged symmetrically and the current in the wires flows uniformly, the electric dipole of the system must remain constant. Hence, radiation arises from the magnetic dipole radiation. Assume the voltage drop due to this radiation as V_r. We write

$$V_r = \frac{P_r(t)}{i(t)} = i(t)R_r = \frac{2}{3}\frac{\ddot{i}^2\ell^4}{4\pi\epsilon_0 c^5 i}$$

where $R_r = \frac{P_r}{i^2}$ is called the radiation resistance of the circuit and $\ddot{i} = \frac{\mathrm{d}^2 i}{\mathrm{d}t^2}$. From our assumption, since R_r is small, the characteristic time τ of the circuit must be large compared to the oscillation period. Thus we approximate as follows:

$$i = i_0 e^{-\frac{t}{\tau}} \cos(\Omega t)$$

$$\frac{\mathrm{d}^2 i}{\mathrm{d}t^2} \approx i_0 e^{-\frac{t}{\tau}} \cos(\Omega t)$$

Thus $\ddot{i} = -\Omega^2 i$. The voltage drop is approximated to

$$V_r = \frac{2}{3}\frac{\Omega^4 i^2 \ell^4}{4\pi\epsilon_0 c^5 i} = \frac{\Omega^4 i \ell^4}{6\pi\epsilon_0 c^5}$$

The dissipated power is given by

$$P_r(t) = \frac{\Omega^4 i^2 \ell^4}{6\pi\epsilon_0 c^5}$$

and

$$R_r = \frac{\Omega^4 \ell^4}{6\pi\epsilon_0 c^5}$$

Noting that $\Omega = \frac{2\pi c}{\lambda}$, we have that $R_r = k\left(\frac{\ell}{\lambda}\right)^4$ where λ is the wavelength of radiation. In practice, R_r is very small.

Solution 2: The net resistance in the system is zero. We have

$$a = \frac{\Delta v}{\tau}$$

Hence, we write the work energy theorem as follows:

$$e\Delta V_{12} = e\left[V_0 - \frac{q}{C_{\text{eq}}}\right] = \frac{1}{2}m\Delta v^2$$

Substituting Δv into Larmor's formula, the power radiated by each charge is

$$\mathrm{d}P_r = \frac{2}{3}\frac{q^2 a^2}{4\pi\epsilon_0 c^3} = \frac{q^2 \Delta v^2}{4\pi\epsilon_0 c^3 \tau^2} = \frac{e^2 \cdot \frac{2e\left[V_0 - \frac{q}{C_{\text{eq}}}\right]}{m}}{4\pi\epsilon_0 c^3 \tau^2}$$

We obtain P_r by integrating the same from $q = 0$ to $q = q_{\text{total}}$.

Problem 6

For Jackson's hypothesis to be true:

$$\frac{1}{2}m_e\Delta v^2 = P_r\Delta\tau = \frac{2}{3}\frac{q^2a^2}{4\pi\epsilon_0 c^3}\tau$$

$$\tau = \frac{e^2}{3\pi\epsilon_0 m_e c^3} \approx \frac{r_e}{3\pi\epsilon_0 c}$$

where r_e is the radius of an electron. Thus, all of the missing energy $\frac{1}{2}C_{\text{eq}}{V_0}^2$ can appear as dipole radiation iff the charges move from one capacitor to the other in the time in which light travels $\frac{1}{3\pi\epsilon_0}$ times the radius of an electron. However, this is theoretically impossible. Thus, Jackson's hypothesis cannot be true.

Problem 7

Let the charge on the capacitor with capacitance C_1 be $q(t)$. At equilibrium, final charge on C_1 is given by $q_0\frac{C_1}{C_1+C_2}$. The potential difference between the plates after the charge has been transferred is given by:

$$\Delta V = \frac{q}{C_1} - \frac{q_0 - q}{C_2}$$

The kinetic energy gained by the charges is then simply

$$\Delta K = \int_{q_0}^{\frac{C_1}{C_1+C_2}q_0} \Delta V \mathrm{d}q = \frac{{q_0}^2 C_2}{2C_1(C_1+C_2)} = \frac{{q_0}^2}{2C_1}\left[\frac{C_2}{C_1+C_2}\right] = \frac{1}{2}C_{\text{eq}}{V_0}^2$$

Problem 8

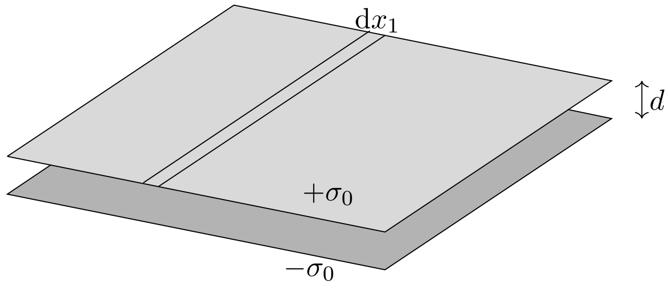
This could be a possible explanation of the missing energy since the energy gained by the charge carriers comes out to be the same as E_Δ. When the charge gets deposited on the plate of the second capacitor, it accumulates there and dissipates thermal energy as it interacts with the charges on the plate of this capacitor.

Grading Scheme

A1	Correct answers for E_∞, E_Δ and $C \to \infty$ case	2.5
A2	Correct analysis of $C_2 \to \infty$ case	0.5
B1	Correct expression for $i(t)$	1.5
B2	Considered underdamping, critical damping and over-damping	0.5
B3	Correct simplified expression of E_Δ	1.0
C1	Correct frequency approximated to the first order	1.0
D1	Noted that all non-zero finite values work	0.5
D2	Second constraint from characteristic time constant	0.5
E1	Correct idea of radiative voltage and resistance	1.0
E2	Correct approximation for large characteristic time constant τ	1.0
E3	Correct calculation of power and energy dissipated	2.0
F1	Correct use of work energy theorem	0.5
F2	Correct final expression for τ	0.5
G1	Exact expression of ΔV	1.0
G2	Correct limits on charge transferred	1.0
G3	Calculation of $\Delta K = \int \Delta V \mathrm{d}q$	1.0
G4	Correct simplified final expression for kinetic energy	1.0
H1	Correct explanation of why the Feynman scenario is possible	1.0

Problem 9

Let us assume that the capacitor plates are lying in the $x-y$ plane, and the electric field between the plates is in the z-direction. The initial potential energy of the system is given as $E_0 = 2\pi\sigma^2 abd$. There are repulsive forces on the charges of the plates parallel to its surface. Usually, these are ignored. But in this case, with the stretching of the plates, work will be performed by these forces. These forces are small near the centre of the plate surface, but as we will find out, appreciable as we go towards the plate edges performing a significant amount of work. The field is thus, fringing.



Since the expansion is done in one dimension, only the x-component of the forces is important. Take a line charge at $x = x_1$ with linear charge density $\sigma \mathrm{d}x_1$. At $x = x_2$ on both plates, we also consider line charges. Net force of repulsion due to both the plates is then given by

$$F_0 = \frac{\sigma^2}{2\pi\epsilon_0}\left[\int_0^{2x_1-a} \frac{\sqrt{b^2+(x_1-x_2)^2}-(x_1-x_2)}{x_1-x_2}\mathrm{d}x_2 \right.$$
$$\left. - \int_0^{2x_1-a} \frac{x_1-x_2}{\sqrt{d^2+(x_1-x_2)^2}}\mathrm{d}x_2 \cdot \frac{b^2+d^2+(x_1-x_2)^2}{\sqrt{d^2+(x_1-x_2)^2}}\sqrt{d^2+(x_1-x_2)^2}\right]$$

Now the plates are expanded quasi-statically such that the following transformations occur $\sigma \to \frac{\sigma}{\kappa}, a \to \kappa a$. As the expansion factor goes from κ to $\kappa + \mathrm{d}\kappa$, the line of charge at x_1 moves a distance $(x - \frac{\kappa a}{2})\frac{\mathrm{d}\kappa}{\kappa}\hat{x}$. The work done by the repulsive forces during expansion of both plates is then

$$\mathrm{d}W = 4 \times \frac{\sigma^2 \mathrm{d}\kappa}{4\pi\epsilon_0\kappa}\int_0^{\kappa a}(2x_1-\kappa a)\cdot(\sqrt{{x_1}^2+b^2}-x_1+\sqrt{{x_1}^2+d^2}-\sqrt{{x_1}^2+b^2+d^2}$$
$$- b\ln\left(\frac{x_1}{\sqrt{{x_1}^2+b^2}-b}\cdot\frac{\sqrt{{x_1}^2+d^2}-b}{\sqrt{{x_1}^2+d^2}}\right)\mathrm{d}x_1$$

Now we use the substitution $\Gamma = x_1 - x_2$, solve the indefinite integrals and simplify our result to obtain:

$$\begin{aligned} \mathrm{d}W = \frac{\sigma^2 \mathrm{d}\kappa}{\pi\epsilon_0\kappa}\Bigg[& \frac{2d^2}{3}\left(\sqrt{\kappa^2a^2+d^2} - d + \sqrt{b^2+d^2} - \sqrt{\kappa^2a^2+b^2+d^2}\right) \\ & -b^2\left(\sqrt{b^2+d^2} + \sqrt{\kappa^2a^2+b^2} - b + \sqrt{\kappa^2a^2+b^2+d^2}\right) \\ & +\frac{\kappa^2a^2}{6}\left(\sqrt{\kappa^2a^2+d^2} + \sqrt{\kappa^2a^2+b^2} - \kappa a - \sqrt{\kappa^2a^2+b^2+d^2}\right) \\ & +\frac{\kappa ab^2}{2}\ln\left(\frac{b}{\sqrt{\kappa^2a^2+b^2}-\kappa a}\cdot\frac{\sqrt{\kappa^2a^2+b^2+d^2}-\kappa a}{\sqrt{b^2+d^2}}\right) \\ & -\frac{\kappa ad^2}{2}\ln\left(\frac{d}{\sqrt{\kappa^2a^2+d^2}-\kappa a}\cdot\frac{\sqrt{\kappa^2a^2+b^2+d^2}-\kappa a}{\sqrt{b^2+d^2}}\right) \\ & -bd^2\ln\left(\frac{d}{\sqrt{d^2+b^2}-b}\cdot\frac{\sqrt{\kappa^2a^2+b^2+d^2}-b}{\sqrt{\kappa^2a^2+d^2}}\right) \\ & +\kappa abd\arctan\left(\frac{\kappa ab}{d\sqrt{\sqrt{\kappa^2a^2+b^2+d^2}}}\right)\Bigg] \end{aligned}$$

This can be written as

$$\mathrm{d}W = 4\sigma^2 abd\frac{\mathrm{d}\kappa}{\kappa^2}\left(\frac{\pi}{2} + \mathcal{O}(\frac{d}{\kappa a}, \frac{d}{b}, \frac{d}{\sqrt{\kappa^2a^2+b^2}})\right)$$

This is our answer for the general case.

Thus, for $d \ll \kappa a, b$, we have

$$\mathrm{d}W = 2\pi\sigma^2 abd\frac{\mathrm{d}\kappa}{\kappa^2} = E_0\frac{\mathrm{d}\kappa}{\kappa^2}$$

Problem 10

There is work done against the repulsive forces between the charges on a plate within each capacitor, and the missing energy is used up as work done against them as the plates are expanded quasi-statically.

This analysis is very similar to what we did in Part A. In this case, there was no thermal, radiative losses or gain in kinetic energy due to the quasi-static nature of the process. Instead, the repulsive forces on the capacitor plates were significant and proportional to the plate dimensions. If some other external agency such as "magic tweezers" had been used to pick up charges from one capacitor to the another quasi-statically, the source of missing energy would have been the work done using the magic tweezers.

Grading Scheme

A1	Realize that repulsive forces between charges on plates are significant	1.0
A2	Write forces between line charges considering both plates	1.5
A3	Correct calculation for the work done in expansion	2.5
A4	Correct final simplified expression of $\mathrm{d}W$ and answer for the $d \ll \kappa a, b$ case	1.0
B1	Correct explanation of work done against repulsive forces	1.0